Problem Set 2
Math Camp, Spring 2020, UCSB
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This problem set will help you review the key concepts from the course so far. Feel free to use any resources from the course or internet and submit them by email to the instructor (before midnight on the due date).

1. Prove 1 - (g).

   - $S \in \mathcal{B}$

$$\begin{aligned}
\mathcal{B} \neq \emptyset \quad &\Rightarrow \quad E \in \mathcal{B} && (\mathcal{B} \text{ is nonempty}) \\
&\Rightarrow \quad E^c \in \mathcal{B} && (\text{closed under complements}) \\
&\Rightarrow \quad E \cup E^c \in \mathcal{B} && (\text{closed under countable unions}) \\
&\Rightarrow \quad S \in \mathcal{B}
\end{aligned}$$

   - $\emptyset \in \mathcal{B}$

$$S \in \mathcal{B} \quad \Rightarrow \quad \emptyset \in \mathcal{B} \qquad (\text{closed under complements})$$

   - $\mathcal{B}$ is closed under countable intersections.

$$\begin{aligned}
E_1, E_2, \cdots \in \mathcal{B} \quad &\Rightarrow \quad E_1^c, E_2^c, \cdots \in \mathcal{B} && (\text{closed under complements}) \\
&\Rightarrow \quad \bigcup_{i=1}^{\infty} E_i^c \in \mathcal{B} && (\text{closed under countable unions}) \\
&\Rightarrow \quad \left(\bigcap_{i=1}^{\infty} E_i \right)^c \in \mathcal{B} && (\text{DeMorgan's Laws}) \\
&\Rightarrow \quad \bigcap_{i=1}^{\infty} E_i \in \mathcal{B} && (\text{closed under complements})
\end{aligned}$$

2. Prove 2 - (c).

- $\mathbb{P}(A^c) = 1 - P(A)$

$$\begin{aligned} \mathbb{P}(S) &= \mathbb{P}(A) + \mathbb{P}(A^c) && (S = A \cup A^c \quad \wedge \quad A \cap A^c = \emptyset) \\ 1 &= \mathbb{P}(A) + \mathbb{P}(A^c) && (\mathbb{P}(S) = 1) \end{aligned}$$

- $\mathbb{P}(A) \leq 1$

$$\begin{aligned} 1 &= \mathbb{P}(A) + \mathbb{P}(A^c) \\ \mathbb{P}(A) &= 1 - \mathbb{P}(A^c) \\ \mathbb{P}(A) &\leq 1 && (\mathbb{P} : \mathcal{B} \mapsto [0, \infty)) \end{aligned}$$

- $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$$\mathbb{P}(B) = \mathbb{P}(B \cap A^c) + \mathbb{P}(B \cap A)$$
$$(B = (B \cap A^c) \cup (B \cap A) \quad \wedge \quad (B \cap A^c) \cap (B \cap A) = \emptyset)$$

- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c) \qquad (A \cup B = A \cup (B \cap A^c) \quad \wedge \quad A \cap (B \cap A^c) = \emptyset)$$

- If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

$$\begin{aligned} \mathbb{P}(B \cap A^c) &= \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ \mathbb{P}(B \cap A^c) &= \mathbb{P}(B) - \mathbb{P}(A) && (A \subseteq B) \\ 0 &\leq \mathbb{P}(B) - \mathbb{P}(A) && (\mathbb{P} : \mathcal{B} \mapsto [0, \infty)) \end{aligned}$$

3. Consider testing for the presence of a disease. The test is very accurate in the sense that if a patient has the disease, the test always comes back positive, i.e., $\mathbb{P}(\text{positive}|\text{disease}) = 1$. Sometimes the test is inaccurate, however, in the sense that the test gives a false positive (a postive value when a person doesn't have the disease) with probability 0.005, i.e., $\mathbb{P}(\text{positive}|\text{no disease}) = 0.005$. If the probability of having the disease is 0.001, i.e., $\mathbb{P}(\text{disease}) = 0.001$, what is the probability a patient has the disease, given they have a positive test?

$$\begin{aligned}\mathbb{P}(\text{positive}) &= \mathbb{P}(\text{disease})\mathbb{P}(\text{positive}|\text{disease}) + \mathbb{P}(\text{no disease})\mathbb{P}(\text{positive}|\text{no disease})\\ &= 0.001 * 1 + 0.999 * 0.005 = 0.005995\end{aligned}$$

Then by Bayes' rule, we have that:

$$\begin{aligned}\mathbb{P}(\text{disease}|\text{positive}) &= \mathbb{P}(\text{positive}|\text{disease})\frac{\mathbb{P}(\text{disease})}{\mathbb{P}(\text{positive})}\\ &= 1 * \frac{0.001}{0.005995} \approx 0.1668\end{aligned}$$

4. A variable X is lognormally distributed if $Y = \ln(X)$ is normally distributed with μ and σ^2, i.e. $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$. Let $x = g(y) = e^y$ and $y = g^{-1}(x) = \ln(x)$.

- Derive $f_X(x)$.

$$f_X(x) = f_Y\left(g^{-1}(x)\right)\left|\frac{dg^{-1}(x)}{dx}\right| = \frac{1}{\sqrt{2\pi\sigma^2}x} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2}$$

- Derive $\mathbb{E}[X^t]$ using $M_Y(t)$. What are $\mathbb{E}[X]$ and $V(X)$?

We know that $M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

$$\begin{aligned}\mathbb{E}[X^t] &= \mathbb{E}[e^{tY}] = M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}\\ \mathbb{E}[X] &= e^{\mu + \frac{1}{2}\sigma^2}\\ \mathbb{E}[X^2] &= e^{2\mu + 2\sigma^2}\\ V(X) &= (e^{\sigma^2} - 1) \cdot e^{2\mu + \sigma^2}\end{aligned}$$

5. Assume that the n units are volunteers to receive the treatment. Given any $i \in \{1, \cdots, n\}$, $D_i = 1$ if treated, and $D_i = 0$ otherwise. Let $(D_1, \cdots, D_n)$ be a vector stacking the treatment indicators of all units. Treatments have capacity constraints and only $n_1 (< n)$ units can be treated: $\sum_{i=1}^{n} D_i = n_1$.

   - What is the number of possible values $(D_1, \cdots, D_n)$ can take?

$$\binom{n}{n_1}$$

   We say that treatment is randomly assigned if $(D_1, \cdots, D_n)$ are random variables, and if for any vector of n numbers $(d_1, \cdots, d_n) \in \{0,1\} \times \cdots \times \{0,1\}$ such that $\sum_{i=1}^{n} d_i = n_1$,

$$P(D_1 = d_1, \cdots, D_n = d_n) = \frac{1}{\binom{n}{n_1}}$$

   That is, random assignment generates uniform treatment probabilities across units.

   - If full randomization is satisfied, then for every $i \in \{1, \cdots, n\}$, what is $P(D_i = 1)$?

$$P(D_i = 1) = \frac{\binom{n-1}{n_1-1}}{\binom{n}{n_1}} = \frac{n_1}{n}$$

   - If full randomization is satisfied, then for every $i \neq j$, what is $P(D_i = 1 \wedge D_j = 1)$? Is it true that unit i getting treated is independent from unit j getting treated?

$$P(D_i = 1 \wedge D_j = 1) = \frac{\binom{n-2}{n_1-2}}{\binom{n}{n_1}} = \frac{n_1}{n} \cdot \frac{n_1 - 1}{n - 1}$$

$$P(D_i = 1 \wedge D_j = 1) \neq P(D_i = 1) P(D_j = 1) \quad \text{and} \quad P(D_i = 1 \mid D_j = 1) \neq P(D_i = 1)$$

   The intuition is that if $D_j = 1$, D_i is less likely to be equal to 1 than if $D_j = 0$: in the first case, there are only $n_1 - 1$ seats for treatment left for $n - 1$ units, while in the second case there are n_1 seats for treatment left for $n - 1$ units.