ECON 204C - Macroeconomic Theory

New Keynesian Model

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Learning Objective

- Log-linearization around the steady state
- New Keynesian model

Log-linearization

$$X_t = X\left(\frac{X_t}{X}\right) = Xe^{\ln(X_t/X)} = Xe^{\ln X_t - \ln X}$$

Taking a first order Taylor approximation around the steady state yields

$$\begin{aligned} Xe^{\ln X_t - \ln X} &\approx Xe^0 + Xe^0\Big[(\ln X_t - \ln X) - 0\Big] \\ &\approx X(1 + \ln X_t - \ln X) \end{aligned}$$

Therefore,

$$X_t \approx X(1 + \ln X_t - \ln X)$$

By the same logic,

$$\begin{aligned} X_tY_t &\approx X(1 + \ln X_t - \ln X)Y(1 + \ln Y_t - \ln Y) \\ &\approx XY(1 + \ln X_t - \ln X + \ln Y_t - \ln Y) \qquad \because \ (\ln X_t - \ln X)(\ln Y_t - \ln Y) \approx 0 \end{aligned}$$

Household UMP

$$\max_{\{\{C_t(i)\}_{i\in[0,1]},\ N_t,\ B_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\phi}}{1+\phi} \right]$$

$$s.t. \quad \int_0^1 P_t(i) C_t(i) di + Q_t B_{t+1} = W_t N_t + B_t + T_t$$

$$C_t = \left(\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$$

Dixit Stiglitz Utility Function

$$\max_{\{C(i)\}_{i\in[0,1]}} \left(\int_0^1 C(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} \quad s.t. \quad \int_0^1 p(i)C(i)di \leq X$$

FOCs are given by

$$\frac{\epsilon}{\epsilon-1}\left(\int_0^1 C(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{1}{\epsilon-1}} \frac{\epsilon-1}{\epsilon} C(j)^{-\frac{1}{\epsilon}} - \lambda p(j) = 0 \quad \forall\ j \in [0,1] \tag{1}$$

$$\int_0^1 P(i)C(i)di = X \tag{2}$$

$$\left(\frac{C(j)}{C(k)}\right)^{\frac{1}{\epsilon}} = \frac{P(k)}{P(j)} \quad \Rightarrow \quad C(j) = \left(\frac{P(k)}{P(j)}\right)^{\epsilon} C(k)$$

Dixit Stiglitz Utility Function

$$X = \int_0^1 P(i)C(i)di = \int_0^1 \left(P(i)\left(\frac{P(k)}{P(i)}\right)^{\epsilon} C(k)\right) di = C(k)P(k)^{\epsilon}\int_0^1 P(i)^{1-\epsilon}di$$

$$C = \left(\int_0^1 C(i)^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}} = \left(\int_0^1 \left\{\left(\frac{P(k)}{P(i)}\right)^{\epsilon} C(k)\right\}^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} = C(k)P(k)^{\epsilon}\left(\int_0^1 P(i)^{1-\epsilon}di\right)^{\frac{\epsilon}{\epsilon-1}}$$

Let $P := \left(\int_0^1 P(i)^{1-\epsilon}di\right)^{\frac{-1}{\epsilon-1}}$. Then $PC = X$.

$$C(k) = \frac{P(k)^{-\epsilon}}{\int_0^1 P(i)^{1-\epsilon}di}X = \frac{P(k)^{-\epsilon}}{P^{1-\epsilon}}X = \left(\frac{P(k)}{P}\right)^{-\epsilon}\frac{X}{P}$$

Household UMP

$$\max_{\{C_t,\ N_t,\ B_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\phi}}{1+\phi} \right]$$

$$s.t. \quad P_t C_t + Q_t B_{t+1} = W_t N_t + B_t + T_t$$

$$C_t = \left(\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t$ and $P_t = \left(\int_0^1 P_t(i)^{1-\epsilon} di \right)^{\frac{-1}{\epsilon-1}}$

$$C_t^{-\gamma} = \lambda_t P_t \qquad \text{(FOC w.r.t. } C_t\text{)}$$

$$-N_t^{\phi} = \lambda_t W_t \qquad \text{(FOC w.r.t. } N_t\text{)}$$

$$\lambda_t Q_t = \beta \mathbb{E}_t[\lambda_{t+1}] \qquad \text{(FOC w.r.t. } B_{t+1}\text{)}$$

Household UMP

$$\underbrace{\frac{N_t^{\phi}}{C_t^{-\gamma}}}_{MRS_t} = \frac{W_t}{P_t} \qquad \text{(Consumption-Leisure)}$$

$$C_t^{-\gamma}\frac{Q_t}{P_t} = \beta\mathbb{E}_t\left[C_{t+1}^{-\gamma}\frac{1}{P_{t+1}}\right] \qquad \text{(Euler equation)}$$

$$w_t - p_t = \phi n_t + \gamma c_t$$

$$1 = \mathbb{E}_t\left[e^{-\rho-\gamma(c_{t+1}-c_t)-\pi_{t+1}+i_t}\right]$$

$$-\ln Q_t = \ln\left(\frac{1}{Q_t} - 1 + 1\right) \approx \frac{1}{Q_t} - 1 = i_t \qquad \text{(Nominal interest rate)}$$

$$-\ln\beta = \ln\left(\frac{1}{\beta} - 1 + 1\right) \approx \frac{1}{\beta} - 1 = \rho \qquad \text{(Utility discount factor)}$$

$$\ln P_{t+1} - \ln P_t = \Delta\ln P_t = \pi_{t+1} \qquad \text{(Inflation rate)}$$

Log-linearization and IS Curve

$$1 = \mathbb{E}_t\left[e^{-\rho-\gamma(c_{t+1}-c_t)-\pi_{t+1}+i_t}\right] \qquad (g_{C,t+1} = c_{t+1} - c_t)$$

$$\approx \mathbb{E}_t\left[e^0 + e^0[i_t - i] - e^0[\pi_{t+1} - \pi] - e^0\gamma[(c_{t+1} - c_t) - g_C]\right] \qquad \text{(Approx. around SS)}$$

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\gamma}\Big(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho\Big) \qquad \text{(At SS, } \rho = i - \pi - g_C\text{)}$$

$$y_t = \mathbb{E}_t[y_{t+1}] - \frac{1}{\gamma}\Big(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho\Big) \qquad \text{(in equilibrium)}$$

$$\underbrace{y_t - y_t^n}_{\tilde{y}_t} = \mathbb{E}_t[\underbrace{y_{t+1} - y_{t+1}^n}_{=\tilde{y}_{t+1}}] - \frac{1}{\gamma}\Big(i_t - \mathbb{E}_t[\pi_{t+1}] - \{\underbrace{\rho + \gamma\mathbb{E}_t[y_{t+1}^n - y_t^n]}_{=r_t^n}\}\Big) \qquad \text{(IS curve)}$$

$$\frac{Q_t}{P_t}R_t = \frac{1}{P_{t+1}} \quad \Leftrightarrow \quad r_t = \underbrace{-\ln Q_t}_{=i_t} - \underbrace{(\ln P_{t+1} - \ln P_t)}_{\pi_{t+1}}$$

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\gamma}\Big(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho\Big) \quad \Leftrightarrow \quad r_t = \rho + \gamma\mathbb{E}_t[y_{t+1} - y_t] \qquad \text{(in equilibrium)}$$

Firm

Production function of a monopolist for good i is given by

$$Y_t(i) = A_t N_{f,t}(i)^{1-\alpha}$$

Taking P_t and C_t as given, a monopolist for good i faces the following demand curve

$$D_t(P) = \left(\frac{P}{P_t}\right)^{-\epsilon} C_t$$

Note here that

$$D_t'(P) = -\epsilon \frac{D_t(P)}{P} \quad \Leftrightarrow \quad \frac{D_t(P)}{D_t'(P)} = -\frac{P}{\epsilon}$$

Taking W_t as given, a monopolist for good i faces the following nominal cost of producing $Y_t(i)$

$$\kappa_t\Big(Y_t(i)\Big) = W_t \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}}$$

Real cost is given by

$$K_t\Big(Y_t(i)\Big) = \frac{W_t}{P_t}\left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} \quad \text{and} \quad K_t'\Big(Y_t(i)\Big) = \frac{W_t}{P_t}\frac{1}{1-\alpha}\left(\frac{Y_t(i)^{\alpha}}{A_t}\right)^{\frac{1}{1-\alpha}}$$

Optimal Price Setting with Calvo Fairy

With probability $1-\theta$, a firm is tapped by Calvo fairy in a given period. With probability θ, a firm must use its previous price.

$$P_t^* = arg\max_P \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \Big[M_{t,t+k} \left\{ P \times D_{t+k}(P) - \kappa_t\big(D_{t+k}(P)\big) \right\} \Big] \text{ where } M_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\gamma} \frac{P_t}{P_{t+k}}$$

Let $\mathcal{M} := \frac{\epsilon}{\epsilon - 1}$. Then FOC is given by

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \Big[M_{t,t+k} \left\{ D_{t+k}(P_t^*) + P_t^* D'_{t+k}(P_t^*) - \kappa'_{+k}\big(D_{t+k}(P_t^*)\big) D'_{t+k}(P_t^*) \right\} \Big] = 0$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[M_{t,t+k} D_{t+k}(P_t^*) \left\{ 1 + P_t^* \frac{D'_{t+k}(P_t^*)}{D_{t+k}(P_t^*)} - \kappa'_{t+k}\big(D_{t+k}(P_t^*)\big) \frac{D'_{t+k}(P_t^*)}{D_{t+k}(P_t^*)} \right\} \right] = 0$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[M_{t,t+k} D_{t+k}(P_t^*) \left\{ 1 - \epsilon + \kappa'_{t+k}\big(D_{t+k}(P_t^*)\big) \frac{\epsilon}{P_t^*} \right\} \right] = 0$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[M_{t,t+k} D_{t+k}(P_t^*) \left\{ P_t^* - \mathcal{M} \kappa'_{t+k}\big(D_{t+k}(P_t^*)\big) \right\} \right] = 0$$

(Multiply P_t^* and devide by $1-\epsilon$)

Optimal Price Setting without rigidities

With probability $1-\theta$, a firm is tapped by Calvo fairy in a given period. With probability θ, a firm must use its previous price.

$$P_t^* = arg\max_P \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \Big[M_{t,t+k} \left\{ P \times D_{t+k}(P) - \kappa_t\big(D_{t+k}(P)\big) \right\} \Big] \text{ where } M_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\gamma} \frac{P_t}{P_{t+k}}$$

When $\theta = 0$,

$$P_t^M = arg\max_P \ P \times D_t(P) - \kappa_t\big(D_t(P)\big)$$

$$\begin{aligned} D_t(P_t^M) + P_t^M D_t'(P_t^M) &= \kappa_t'\big(D_t(P_t^M)\big) D_t'(P_t^M) \\ P_t^M &= \underbrace{\left(\frac{\epsilon}{\epsilon - 1} \right)}_{=\mathcal{M}} \kappa_t'\big(D_t(P_t^M)\big) \\ \frac{1}{\mathcal{M}} &= \frac{\kappa_t'\big(D_t(P_t^M)\big)}{P_t^M} \qquad \text{(Real MC when } \theta = 0\text{)} \end{aligned}$$

$\mathcal{M}$ is referred to as the desired or frictionless markup.

Optimal Price Setting with Calvo Fairy and Log-linearization

Let $\Pi_{t-1,t+k} := \frac{P_{t+k}}{P_{t-1}}$. Then

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[\beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\gamma} \frac{P_t}{P_{t+k}} D_{t+k}(P_t^*) \left\{ P_t^* - \mathcal{M}\kappa'_{t+k}\big(D_{t+k}(P_t^*)\big) \right\} \right] = 0$$

$$\sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[\left(\frac{C_{t+k}}{C_t} \right)^{-\gamma} \frac{P_t}{P_{t+k}} D_{t+k}(P_t^*) \left\{ \frac{P_t^*}{P_{t-1}} - \mathcal{M}K'_{t+k}\big(D_{t+k}(P_t^*)\big)\Pi_{t-1,t+k} \right\} \right] = 0$$

In the zero inflation steady state, $P_t^* = P_{t-1}$, $\Pi_{t-1,t+k} = 1$, $P_t^* = P_{t+k}$, $D_{t+k}(P_t^*) = D$, $K'_{t+k}\big(D_{t+k}(P_t^*)\big) = MC$, and $C_{t+k} = C$. Accordingly, in the steady state, $MC = \frac{1}{\mathcal{M}}$.

$$\sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[\left(\frac{C_{t+k}}{C_t} \right)^{-\gamma} \frac{P_t}{P_{t+k}} D_{t+k}(P_t^*) \frac{P_t^*}{P_{t-1}} \right] = \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[\left(\frac{C_{t+k}}{C_t} \right)^{-\gamma} \frac{P_t}{P_{t+k}} D_{t+k}(P_t^*) \mathcal{M}K'_{t+k}(D_{t+k}(P_t^*))\Pi_{t-1,t+k} \right]$$

Let $\widehat{mc}_{t+k|t} := -\ln\left(\frac{1}{\mathcal{M}}\right)$. Log-linearize around the steady state, then

$$\frac{1}{1-\beta\theta}[p_t^* - p_{t-1}] = \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[\widehat{mc}_{t+k|t} + [p_{t+k} - p_{t-1}] \right]$$

$$p_t^* - p_{t-1} = \mu + (1-\beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[mc_{t+k|t} + [p_{t+k} - p_{t-1}] \right]$$

Aggregate Price Dynamics and Log-linearization

Let $S(t) \subset [0,1]$ represent the set of firms not reoptimizing their posted price in period t.

$$P_t = \left[\int_{S(t)} P_{t-1}(i)^{1-\epsilon} di + (1-\theta) P_t^{*\,1-\epsilon} \right]^{\frac{1}{1-\epsilon}}$$

$$= \left[\theta P_{t-1}^{1-\epsilon} + (1-\theta) P_t^{*\,1-\epsilon} \right]^{\frac{1}{1-\epsilon}}$$

The distribution of prices among firms not adjusting in period t corresponds to the distribution of effective prices in period $t-1$, though with total mass reduced to θ. Divide both sides by P_{t-1}, then

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta) \left(\frac{P_t^*}{P_{t-1}} \right)^{1-\epsilon}$$

Log-linearize around zero inflation steady state, then

$$(1-\epsilon)\pi_t = (1-\theta)(1-\epsilon)(p_t^* - p_{t-1})$$

$$\pi_t = (1-\theta)(p_t^* - p_{t-1})$$

Reference

Gali, J. (2015). Monetary policy, inflation, and the business cycle: an introduction to the new Keynesian framework and its applications. Princeton University Press.