ECON 204C - Macroeconomic Theory

Asset Pricing

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Learning Objective

- Asset Pricing
  - Law of iterated expectation
  - Representative agent consumption-based asset pricing
  - CRRA and risk premium
  - Elasticity of Substitution

Law of Iterated Expectation (LIE)

$$\mathbb{E}\Big[\ \mathbb{E}\big[Y|X,Z\big]\ |\ X\ \Big] = \mathbb{E}[Y|X]$$

For given $\tau \geq 2$,

$$\mathbb{E}_t[X_{t+\tau}] = \mathbb{E}[X_{t+\tau}|\mathcal{I}_t]$$
$$\mathbb{E}_{t+1}[X_{t+\tau}] = \mathbb{E}[X_{t+\tau}|\mathcal{I}_{t+1}]$$
$$\mathcal{I}_t \subset \mathcal{I}_{t+1}$$
$$\mathbb{E}_t\Big[\ \mathbb{E}_{t+1}\big[X_{t+\tau}\big]\Big] = \mathbb{E}_t[X_{t+\tau}]$$

The smaller information set wins!

Representative Agent Consumption-based Asset Pricing

$$\max_{\{c_t,\ \{a_{j,t+1}\}_j\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$s.t. \quad c_t + \sum_j p_{j,t} a_{j,t+1} = y_t + \sum_j (p_{j,t} + d_{j,t}) a_{j,t}$$

$$\underbrace{p_{j,t} u'(c_t)}_{MC} = \underbrace{\mathbb{E}_t \left[\beta u'(c_{t+1}) [\ p_{j,t+1} + d_{j,t+1}\ ] \right]}_{MB} \qquad \text{(Euler equation)}$$

$$p_{j,t} = \mathbb{E}_t \left[\underbrace{\beta \frac{u'(c_{t+1})}{u'(c_t)}}_{\text{SDF}} [\ p_{j,t+1} + d_{j,t+1}\ ] \right] \qquad \text{(Asset Pricing)}$$

$$p_{j,t} = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} [\ p_{j,t+1} + d_{j,t+1}\ ] \right] \qquad \text{(CRRA)}$$

Representative Agent Consumption-based Asset Pricing

$$
\begin{aligned}
p_{j,t} &= \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left[\,p_{j,t+1}+d_{j,t+1}\,\right]\right] && \text{(Asset Pricing)}\\
&= \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} p_{j,t+1}\right] + \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} d_{j,t+1}\right]\\
&= \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \mathbb{E}_{t+1}\left[\beta\left(\frac{c_{t+2}}{c_{t+1}}\right)^{-\gamma}\left[\,p_{j,t+2}+d_{j,t+2}\,\right]\right]\right] + \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} d_{j,t+1}\right] && \text{(Recursion)}\\
&= \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\beta\left(\frac{\cancel{c_{t+1}}}{c_t}\right)^{-\gamma} \beta\left(\frac{c_{t+2}}{\cancel{c_{t+1}}}\right)^{-\gamma}\left[\,p_{j,t+2}+d_{j,t+2}\,\right]\right]\right] + \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} d_{j,t+1}\right] && \text{(Information)}\\
&= \mathbb{E}_t\left[\beta^2\left(\frac{c_{t+2}}{c_t}\right)^{-\gamma}\left[\,p_{j,t+2}+d_{j,t+2}\,\right]\right] + \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} d_{j,t+1}\right] && \text{(LIE)}\\
&= \mathbb{E}_t\left[\beta^2\left(\frac{c_{t+2}}{c_t}\right)^{-\gamma} p_{j,t+2}\right] + \sum_{\tau=1}^{2}\mathbb{E}_t\left[\beta^{\tau}\left(\frac{c_{t+\tau}}{c_t}\right)^{-\gamma} d_{j,t+\tau}\right]\\
&= \cdots\\
&= \mathbb{E}_t\left[\lim_{k\to\infty}\beta^{k}\left(\frac{c_{t+k}}{c_t}\right)^{-\gamma} p_{j,t+k}\right] + \sum_{\tau=1}^{\infty}\mathbb{E}_t\left[\beta^{\tau}\left(\frac{c_{t+\tau}}{c_t}\right)^{-\gamma} d_{j,t+\tau}\right]\\
&= \sum_{\tau=1}^{\infty}\mathbb{E}_t\left[\beta^{\tau}\left(\frac{c_{t+\tau}}{c_t}\right)^{-\gamma} d_{j,t+\tau}\right] && \text{(TVC)}\\
\frac{p_{j,t}}{d_{j,t}} &= \sum_{\tau=1}^{\infty}\mathbb{E}_t\left[\beta^{\tau}\left(\frac{c_{t+\tau}}{c_t}\right)^{-\gamma} \frac{d_{j,t+\tau}}{d_{j,t}}\right] && \text{(Divide by } d_{j,t}\text{)}
\end{aligned}
$$

Constant Relative Risk Aversion and Risk Premium

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad \gamma = -\frac{u''(c)}{u'(c)}c$$

Consider offering two alternatives to a consumer who starts off with risk-free consumption level c_0.

She can receive

1. $c_0 + \Delta_c$ with certainty, or

$$L_1(\Delta_c) = (c_0 + \Delta_c : 1)$$

2. a lottery paying $c_0 - y$ with probability .5 and $c_0 + y$ with probability .5

$$L_2 = (c_0 - y, c_0 + y : .5, .5)$$

Constant Relative Risk Aversion and Risk Premium

For given values of y and c_0, we want to find $\Delta_c = \Delta_c(y, c_0)$ that leaves the consumer indifferent between these two lotteries.

$$L_1(\Delta_c(y, c_0)) \sim L_2$$
$$u\Big(c_0 - \Delta_c(y, c_0)\Big) = .5u\Big(c_0 - y\Big) + .5u\Big(c_0 + y\Big)$$

Taylor expansion of u around the point a

$$u(x) \approx u(a) + u'(a)(x - a) \qquad \text{(First order approximation)}$$
$$u(x) \approx u(a) + u'(a)(x - a) + \frac{1}{2}u''(a)(x - a)^2 \qquad \text{(Second order approximation)}$$

Constant Relative Risk Aversion and Risk Premium

$$u\Big(c_0 - \Delta_c(y, c_0)\Big) = .5u\Big(c_0 - y\Big) + .5u\Big(c_0 + y\Big)$$

First order approximate LHS around c_0.

$$u\Big(c_0 - \Delta_c(y, c_0)\Big) \approx u(c_0) - u'(c_0)\Delta_c(y, c_0)$$

Let Y denote random variable taking value y with probability .5 and $-y$ with probability .5. Second order approximate RHS around c_0.

$$u(c_0 + Y) \approx u(c_0) + u'(c_0)Y + \frac{1}{2}u''(c_0)Y^2$$

$$\mathbb{E}[u(c_0 + Y)] \approx u(c_0) + \frac{1}{2}u''(c_0) \underbrace{y^2}_{=Var(Y)}$$

Constant Relative Risk Aversion and Risk Premium

Ignoring the higher-order terms gives

$$\Delta_c(y, c_0) \approx \frac{1}{2} y^2 \left(-\frac{u''(c_0)}{u'(c_0)} \right)$$

$$\Delta_c(y, c_0) \approx \frac{1}{2} y^2 \frac{\gamma}{c_0} \qquad \text{(CRRA)}$$

$$\frac{\Delta_c(y, c_0)}{y} \approx \frac{1}{2} \gamma \frac{y}{c_0}$$

- LHS is the percentage premium that the consumer is willing to pay to avoid a fair bet of size y
- RHS is one half γ times the ratio of the size of the bet y to her initial consumption level c_0.

Constant Relative Risk Aversion and Risk Premium

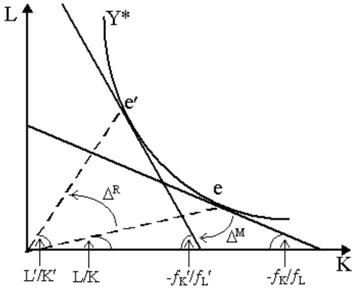
Following Cochrane (1997), think of confronting someone with initial consumption of \$50,000 per year with a 50 - 50 chance of winning or losing \$$y$.

$\gamma \setminus y$	10	100	1000	5000
2	.02	.2	20	500
5	.05	5	50	1217
10	.1	1	100	2212

Risk Premium $\Delta_c(y, c_0 = 50000; \gamma)$

For values of γ even as high as 5, risk premiums are too big. This result is one important source of macroeconomists' prejudice that γ should not be much higher than 2 or 3.

Elasticity of Substitution



$$\Delta M = \Delta\left(-\frac{dL}{dK}\Big|_{Y^*=F(K^*,L^*)}\right) = \Delta\left(\frac{F_K(K^*,L^*)}{F_L(K^*,L^*)}\right)$$

$$\Delta R = \Delta\left(\frac{L}{K}\right)$$

$$\text{Elasticity} = \frac{\frac{d(L/K)}{(L/K)}}{\frac{d(F_K/F_L)}{(F_K/F_L}} = \frac{d\ln(L/K)}{d\ln(F_K/F_L)}$$

Constant Elasticity of Substitution (CES)

$$Y = F(K, L) = \left(\alpha_L L^{\frac{\varepsilon-1}{\varepsilon}} + \alpha_K K^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$MRTS_{L,K} = -\frac{dL}{dK}\bigg|_{Y=F(K,L)} = \frac{F_K}{F_L} = \frac{\alpha_K}{\alpha_L}\left(\frac{L}{K}\right)^{\frac{1}{\varepsilon}}$$

$$\frac{L}{K} = \left(\frac{\alpha_L}{\alpha_K}\right)^{\varepsilon}\left(\frac{F_K}{F_L}\right)^{\varepsilon}$$

$$\ln(L/K) = \varepsilon \ln(\alpha_L/\alpha_K) + \varepsilon \ln(F_K/F_L)$$

$$\frac{d\ln(L/K)}{d\ln(F_K/F_L)} = \varepsilon$$

CRRA and EIS

$$U = U(c_1, c_2) = u(c_1) + \beta u(c_2) \qquad \text{where} \qquad u(x) = \frac{x^{1-\gamma}}{1-\gamma}$$

$$\begin{aligned}
MRS_{2,1} &= -\frac{dc_2}{dc_1}\bigg|_{U=U(c_1,c_2)} = \frac{U_1}{U_2} = \frac{u'(c_1)}{\beta u'(c_2)} = \frac{c_1^{-\gamma}}{\beta c_2^{-\gamma}} = \frac{1}{\beta}\left(\frac{c_1}{c_2}\right)^{-\gamma} \\
\frac{c_2}{c_1} &= \beta^{1/\gamma}\left(\frac{U_1}{U_2}\right)^{1/\gamma} \\
\ln(c_2/c_1) &= \frac{1}{\gamma}\ln\beta + \frac{1}{\gamma}\ln(U_1/U_2) \\
\frac{d\ln(c_2/c_1)}{d\ln(U_1/U_2)} &= \frac{1}{\gamma}
\end{aligned}$$

Reference

Ljungqvist, L., & Sargent, T. J. (2018). Recursive macroeconomic theory. MIT press.