ECON 204C - Macroeconomic Theory

Dynamic programming review and computation

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General Information

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- Zoom Office Hours: Tuesdays and Thursdays 08:00 - 09:00 AM PDT
- Problem sets need to be done in LaTeX.
- I will be using Python with Jupyter as my editor.
  You can use any language that you would like.

Learning Objective

- Dynamic programming in discrete time
  - Stochastic optimal growth model
  - Bellman equation / Bellman operator / Policy function
  - Contraction mapping theorem

- Computation
  - Interpolation
  - Integration
  - Minimization
  - Value function iteration

Preferences

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

- u is a bounded, continuous and strictly increasing utility function.

$$u(c_t) = \ln\, c_t$$

- $\beta = 0.96 \in (0,1)$ is a discount factor.

Resource Constraints

$$y_t \geq c_t + [\ \underbrace{k_{t+1} - (1 - \delta)k_t}_{=i_t}\ ] \quad \text{where} \quad \delta = 1$$

- An agent owns an amount $y_t \in \mathbb{R}_+ := [0, \infty)$ of consumption good at time t.
- Output can either be consumed or invested.
- When the good is invested, it is transformed one-for-one into capital.

Technology

$$y_{t+1} = f(k_{t+1})\xi_{t+1} \qquad \text{where} \qquad \xi_{t+1} \sim \Phi$$

- $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is called the production function which is increasing and continuous in k.

$$f(k_{t+1}) = k_{t+1}^{\alpha} \qquad \text{where} \qquad \alpha = 0.4 \in (0,1)$$

- Production is stochastic, in that it depends on a shock ξ_{t+1} realized at the end of the current period t.

$$\xi_{t+1} = \exp\{\mu + s\zeta_{t+1}\} \qquad \text{where} \qquad \zeta_{t+1} \overset{i.i.d.}{\sim} \mathcal{N}(0,1), \qquad \mu = 0, \qquad s = 0.1$$

Optimization

$$\max_{\{c_t,\ k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

$$s.t. \quad y_t \geq c_t + k_{t+1} \quad \forall t \qquad \text{(Resource Constraint)}$$

$$y_{t+1} = f(k_{t+1})\xi_{t+1} \quad \text{where} \quad \xi_{t+1} \overset{i.i.d.}{\sim} \Phi \quad \forall t \qquad \text{(Technology)}$$

$$c_t \geq 0 \quad k_{t+1} \geq 0 \quad \forall t \qquad \text{(Non-negativity Constraint)}$$

$$y_0 = \bar{y}_0 \quad \text{given}$$

Sequential Problem (SP)

$$
\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]
$$

$$
\begin{aligned}
s.t. \quad & y_{t+1} = f(\underbrace{y_t - c_t}_{=k_{t+1}})\xi_{t+1} \quad \text{where} \quad \xi_{t+1} \overset{i.i.d.}{\sim} \Phi \quad \forall t \\
& y_t \geq c_t \geq 0 \quad \forall t \\
& y_0 = \bar{y}_0 \quad \text{given}
\end{aligned}
$$

- Resource constraint holds with equality because u is strictly increasing.
- y_t **summarizes** the state of the world at the start of each period.
- c_t is a value **chosen** by the agent each period after observing the state.

Funtional Equation (FE)

(SP) is an infinite-dimensional optimization problem. We need to find an optimal infinite sequence $\{c_t^*\}_{t=0}^{\infty}$ solving (SP). Dynamic programming (DP) helps us find a simpler maximization problem.

$$v^*(y) = \max_{c \in [0,y]} \left\{ u(c) + \beta \int v^* \Big(f(y-c)\xi \Big) \phi(d\xi) \right\} \qquad \text{(Bellman equation)}$$

Solution v^*, evaluated at $y = \bar{y}_0$, gives the value of the maximum in (SP).

Bellman Operator

An operator is a map that sends functions into functions. Bellman operator T maps a function $w : \mathbb{R} \to \mathbb{R}$ into another function such that given any arbitrary $y \in \mathbb{R}$,

$$Tw(y) := \max_{c \in [0,y]} \left\{ u(c) + \beta \int w\Big(f(y-c)\xi\Big)\phi(d\xi) \right\}$$

Notice that T is a contraction mapping with modulus β (Blackwell's sufficient conditions for a contraction).

When $w : \mathbb{R} \to \mathbb{R}$ is taken to be the value function, w-greedy policy σ optimally trades off current and future rewards such that given any arbitraty $y \in \mathbb{R}$,

$$\sigma(y) := arg \max_{c \in [0,y]} \left\{ u(c) + \beta \int w\Big(f(y-c)\xi\Big)\phi(d\xi) \right\}$$

Contraction Mapping Theorem

If T is a contraction mapping with modulus β, then

1. there exists a unique fixed point v^*, and

$$v^* = Tv^* \qquad \text{or equivalently} \qquad v^*(y) = \max_{c\in[0,y]} \left\{ u(c) + \beta \int v^*\Big(f(y-c)\xi\Big)\phi(d\xi) \right\} \ \forall y \in \mathbb{R}$$

2. for any v_0 and any $n \in \mathbb{N}$,

$$d(T^n v_0, v^*) \leq \beta^n d(v_0, v^*)$$

Associated optimal policy function is given by

$$\sigma^*(y) = arg \max_{c\in[0,y]} \left\{ u(c) + \beta \int v^*\Big(f(y-c)\xi\Big)\phi(d\xi) \right\} \ \forall y \in \mathbb{R}$$

Computation

How can we implement Bellman operator on our computer?

$$
Tw(y) := \underbrace{\max_{c \in [0,y]} \left\{ u(c) + \beta \underbrace{\int \underbrace{w\Big(f(y-c)\xi\Big)}_{\text{1. Approximation}} \phi(d\xi)}_{\text{2. Integration}} \right\}}_{\text{3. Optimization}}
$$

Approximation

- Approximate an analytically intractable real-valued function f with a computationally tractable function $\widehat{f}$ given limited information about f.

- Divide the **approximation domain** of the function into **finite number of sub-intervals** and approaximate the original function in each of the intervals **using some simple functions**, like polynomials.
  - The points on the domain which separate the intervals are called grid points.
  - We use the value of the function at each grid point to approximate the original function.

Linear Interpolation

Given $(x_0, f(x_0))$ and $(x_1, f(x_1))$, fit a linear interpolant through the data.

$$\widehat{f}(x) = b_0 + b_1(x - x_0)$$

At $x = x_0$,

$$\begin{aligned}\widehat{f}(x_0) &= f(x_0)\\ b_0 &= f(x_0)\end{aligned}$$

At $x = x_1$,

$$\begin{aligned}\widehat{f}(x_1) &= f(x_1)\\ b_0 + b_1(x_1 - x_0) &= f(x_0)\\ b_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0}\end{aligned}$$

Linear Interpolation - Application

In order to figure out Bellman operator, we need to approximate an analytically intractable real-valued function w.

$$Tw(y) := \max_{c \in [0,y]} \left\{ u(c) + \beta \int w\Big(f(y-c)\xi \Big) \phi(d\xi) \right\}$$

1. Determine an approximation domain of w.
2. Divide the approximation domain into finite number of sub-intervals.
3. Approximate the original function w in each of the intervals using linear spline.

Monte Carlo Integration

- A random variable X is distributed with *pdf* f_X.
- Consider a function $g : \mathbb{R} \to \mathbb{R}$. Given a random sample of size n, $\{X_i\}_{i=1}^n$,

$$\frac{1}{n}\sum_{i=1}^{n} g(X_i) \quad \xrightarrow{p} \quad \int g(x) f_X(x) dx \qquad \text{(LLN)}$$

- Consider another function $h : \mathbb{R} \to \mathbb{R}$. We need to evaluate $I = \int h(x)dx$. Let us define $g(x) = \frac{h(x)}{f_X(x)}$. Then

$$I = \int h(x)dx = \int \underbrace{\frac{h(x)}{f_X(x)}}_{=g(x)} f_X(x)dx$$

  Therefore, I can be estimated with

$$\frac{1}{n}\sum_{i=1}^{n} \frac{h(X_i)}{f_X(X_i)} \quad \xrightarrow{p} \quad \int h(x)dx$$

Monte Carlo Integration - Application

In order to figure out Bellman operator, we need to evaluate continuation value.

$$Tw(y) := \max_{c \in [0,y]} \left\{ u(c) + \beta \int w\Big(f(y-c)\xi\Big)\phi(d\xi) \right\}$$

Given a random sample of size n, $\{\xi_i\}_{i=1}^n$

$$\frac{1}{n}\sum_{i=1}^{n} \frac{w\Big(f(y-c)\xi_i\Big)\cancel{\phi(\xi_i)}}{\cancel{\phi(\xi_i)}} \qquad \xrightarrow{p} \qquad \int w\Big(f(y-c)\xi\Big)\phi(\xi)d\xi$$

Optimization

- Find the minimum of some real-valued function of several real variables on a domain that has been specified.
  - Golden section search, Newton's method, etc.

- Finding the global minimum can be challenging.
  - The function can have many local minima.
  - More difficulties are to be expected in handling functions of many variables having a number of constraints.

Optimization - Application

In order to figure out Bellman operator, we need to maximize over feasible consumption set. For any $y \in \mathbb{R}$,

$$Tw(y) := \max_{c \in [0,y]} \left\{ u(c) + \beta \int w\Big(f(y-c)\xi\Big)\phi(d\xi) \right\}$$

Reference

Judd, K. L. (1998). Numerical methods in economics. MIT press.

Sargent, T. S. & Stachurski, J. (2020, March 30). Optimal growth I: The stochastic optimal growth model. Retrieved from https://python-intro.quantecon.org/optgrowth.html